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Parametric controllability of certain systems of rigid bodies☆

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Abstract

The concept of parametric controllability as applied to systems of rigid bodies is discussed. The topic of discussion is Langrangian systems for which "unfreezing" of the parameters is possible such as, for example, the refinement of a model by taking account of the small variability of the links assumed by rigid bodies in the first approximation. As has been shown, just taking account of a small change in the parameters can ensure the controllability of a mechanism which was not controllable assuming rigidity absolute of the links. Certain sufficient conditions are proposed for parametric controllability in invariant manifolds for objects with cyclic coordinates. A two-link pendulum in the horizontal plane under the action of an internal moment (from the first link to the second) is considered as an example. The effect of its mass inertial parameters on the controllability is investigated. The parametric controllability of such an object in a manifold of zero angular momentum, due to the elastic pliability of the second link or the oscillations of an additional mass on a spring along the second link, is demonstrated. An example of a parametrically controllable planetary mechanism with slippage of discs is also considered.

When modelling mechanical systems, the question arises of the effect of the degrees of freedom which are neglected on the form of the motion. Examples are known when taking account of small elastic pliability in the links, a "small amount of free play" in the construction of the links or slight variability of the rigid bodies (when the parameters are "unfrozen") leads to qualitative differences in the stability properties of the motion. The effect of such factors on the controllability, which is understood in the traditional sense,[1] is investigated below.

1. Auxiliary results

Natural Lagrangian systems are considered, that is, objects with a Lagrangian function

$$L\Big((\mathbf{q},\dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{A}(\mathbf{q})\dot{\mathbf{q}} - B(\mathbf{q})\Big)$$

which is symmetric with respect to time reversal ($t \to -t$), where $\mathbf{q} = (q_1, q_2 \ldots, q_n)^T$ is the vector of the generalized coordinates and $A(\mathbf{q})$ is a positive definite inertia matrix. We will assume that the potential energy $B(\mathbf{q})$ has a lower

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limit: $B(\mathbf{q}) \geq 0$, $B(\mathbf{0}) = 0$ and that the equations of motion have the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = \mathbf{u} \tag{1.1}$$

The control vector $\mathbf{u} = (u_1, \ldots, u_n)^T$ satisfies the constraints $|u_i| \leq a_i$, where a_i are prespecified numbers ($i = 1, 2, \ldots, n$). In a special case, some of the quantities a_i can be zero, that is, the number of degrees of freedom n can exceed the number of controls. For example, if $a_i = 0$ ($i = 1, 2, \ldots, n$, $i \neq j$), then system (1.1) "is controlled using a scalar input u_j". Since the Lagrangian function can be periodic with respect to certain ("angular") coordinates q_i (suppose there are r of them), we consider a configurational space $M = \mathbf{T}^r \times \mathbf{R}^{n-r}$, where $\mathbf{T}^r$ is an r-dimensional torus. Then the phase space $\mathrm{TM} = \mathbf{T}^r \times \mathbf{R}^{2n-r}$. The notation $(\mathbf{q}, \dot{\mathbf{q}}) \in \mathrm{TM}$ implies that the numerical values of the coordinates are taken from the corresponding covering space $\mathbf{R}^r \times \mathbf{R}^{2n-r}$.

If the separatrix surface in TM, along which motion to a singular point occurs after a finite time, corresponds to the feedback $\mathbf{u} = \mathbf{u}(\mathbf{q}, \dot{\mathbf{q}})$, then the surface is denoted by $\Omega(\mathbf{u}(q, \dot{\mathbf{q}}))$. The set of equilibrium positions $\zeta_0 = \{(\mathbf{q}, \dot{\mathbf{q}}) : \dot{\mathbf{q}} = \mathbf{0}, \partial B/\partial \mathbf{q} = \mathbf{0}, \mathbf{u} = \mathbf{0}\}$ is non-empty, and we assume that the number of components of this manifold is finite.

The number of degrees of freedom can exceed the number of controlling actions. We shall rely on the results in Refs. 2–4 in which the sufficient conditions for global controllability (that is, in the whole of the phase space TM) were based on the symmetry properties of the equations with respect to time reversal and the local controllability in the neighbourhoods of the equilibrium positions or steady rotation. Moreover, the stabilizability to these positions was used, that is, the possibility of transferring the system from each point of phase space to the neighbourhood of a required state, which may be as small as desired. The direct Lyapunov method in the theory of stability was invoked to prove the stabilizability. The concept of a strong Lyapunov function was introduced in Ref. 6 in order to extend the well-known Barbashin–Krasovskii theorem[5] to the case of a cylindrical phase space, which is characteristic of pendulum mechanisms. The traditional definitions of stabilizability and local and global controllability have been indicated in Ref. 3 and, in the case of time-reversible systems ($t \to -t$), global controllability is equivalent to null-controllability.[1]

We will formulate one of the results in Ref. 2 in the following form.

Assertion 1. Suppose that, in system (1.1), the potential $B(\mathbf{q})$ is a strong Lyapunov function in M and that the sets, specified by the conditions $B(\mathbf{q}) \leq c$ ($c > 0$), are compact. Then,

1) if, in the case of free motion ($\mathbf{u} \equiv \mathbf{0}$), the system does not admit of a particular solution $\dot{q}_j \equiv 0$ (excluding equilibrium positions), then it is stabilizable with respect to an input $u_j (j \in 1, 2, \ldots, n)$ in the manifold $\mathrm{TM} \backslash \Omega \cdot (u_j)$.
2) if, at the same time, local controllability with respect to the same input u_j holds in the neighbourhoods of all points $(\mathbf{q}_0, \mathbf{0}) \in \xi_0$, then system (1.1) is globally controllable solely under the action of u_j.

2. Preliminary discussions

We will now consider a classical example of a controllable system. Suppose a planar two-link pendulum outside a gravitational field (Fig. 1) is modelled by weightless rigid rods of lengths l_1 and l_2 with point masses m_1 and m_2 on the ends. The position of the system is specified by the angles φ (measured from the central axis) and ψ (with respect

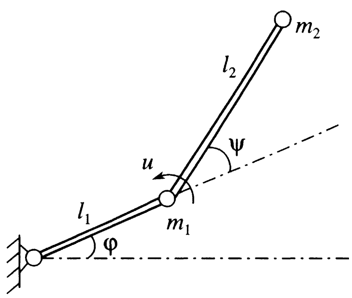

Fig. 1.

to the first rod), that is, $\mathbf{q} = (\varphi, \psi)^T$, $\mathbf{q} \in T^2$. The internal control moment u is bounded in front by a specified quantity: $|u| \le a$.

Using the dimensionless parameters and dimensionless time

$$\mu = \frac{m_1}{m_2}, \quad \lambda = \frac{l_1}{l_2}, \quad a' = \frac{a}{m_2 g l_2}, \quad u' = \frac{u}{m_2 g l_2}, \quad t' = t\sqrt{\frac{g}{l_2}}$$

we obtain the reduced Lagrangian function and equation of motion (we omit the primes)

$$L = \frac{1}{2}\alpha_{11}(\psi)\dot{\varphi}^2 + \frac{1}{2}\alpha_{22}\dot{\psi}^2 + \alpha_{12}(\psi)\dot{\varphi}\dot{\psi}$$

$$\dot{p}_1 = 0 \quad (1 + \lambda\cos\psi)\ddot{\varphi} + \ddot{\psi} + \lambda\dot{\varphi}^2\sin\psi = u, \quad |u| \le a$$

Here,

$$\alpha_{11}(\psi) = (\mu + 1)\lambda^2 + 1 + 2\lambda\cos\psi, \quad \alpha_{12}(\psi) = 1 + \lambda\cos\psi, \quad \alpha_{22} = 1$$

The first equation of motion gives the law of conservation of angular momentum

$$p_1 = \alpha_{11}(\psi)\dot{\varphi} + \alpha_{12}(\psi)\dot{\psi} = \text{const}$$

about the axis of rotation of the first rod.

The system is not globally controllable (in the whole of the phase space $\text{TM} = \mathbf{T}^2 \times \mathbf{R}^2$) since combinations of ψ, $\dot{\psi}$, $\dot{\varphi}$, which endow the function p_1 with another value are known to be inaccessible.

We now consider the important practical case when $p_1 = 0$, which corresponds to an initial (or final) position of rest. Then, for any control actions $u(t)$, the system remains in the manifold

$$\Gamma = \{(\mathbf{q}, \dot{\mathbf{q}}): 2\lambda(b + \cos\psi)\dot{\varphi} + (1 + \lambda\cos\psi)\dot{\psi} = 0\} \tag{2.1}$$

where

$$b = [(1 + \mu)\lambda^2 + 1]/(2\lambda) > (\lambda^2 + 1)/(2\lambda) \ge 1$$

The equation in expression (2.1) is integrated in the form

$$\varphi - \varphi_0 = -\frac{1}{2\lambda}\int_{\psi_0}^{\psi}\frac{1 + \lambda\cos\theta}{b + \cos\theta}d\theta = \Phi(\psi) - \Phi(\psi_0) \tag{2.2}$$

where φ_0 and ψ_0 are the initial values of the coordinates. The function

$$\Phi(\psi) = -\frac{1}{2}\psi + \frac{\lambda b - 1}{\lambda\sqrt{b^2 - 1}}\text{arctg}\left[\sqrt{\frac{b-1}{b+1}}\text{tg}\frac{\psi}{2}\right] \tag{2.3}$$

subject to the condition $\lambda < 1$, is a monotonically increasing function. In the torus $\varphi \times \psi$, the form of the curve (2.2) is determined by the "increase in the angles per period". When $\psi_1 - \psi_0 = 2\pi$, we obtain

$$\varphi_1 - \varphi_0 = \Phi(2\pi) - \Phi(0) = \left(\frac{\lambda b - 1}{\lambda\sqrt{b^2 - 1}} - 1\right)\pi = k\pi \tag{2.4}$$

If the number k is rational, the curve in the torus is closed; otherwise it fills the torus compactly everywhere.

Every curve (2.2) from the manifold (2.1) is determined by the initial coordinates (φ_0, ψ_0). Only those configurations (φ_f, ψ_f) which also satisfy relation (2.2) are accessible in the manifold Γ from this position.

Remark 1. A purposeful motion along the curve (2.2) is also feasible in the case of an additional condition of optimality in the speed of response. It can be shown that, in the $(\psi, \dot{\psi})$ plane, the phase-plane diagram of the optimal synthesis is homeomorphic with the well-known case $\ddot{y} = v$.[7] The optimal motion consists of an "acceleration" (from the initial state $(\psi_0, \dot{\psi}_0)$) and a "retardation" along the line of switching to the state $(\psi_f, 0)$. Then, at the end of the motion (by virtue of the fact that $p_1 = 0$), we also obtain $\dot{\varphi} = 0$ and $\varphi = \varphi_f$.

Actually, when account is taken of the equality $p_1 = 0$, the energy of the system can be represented in the form

$$E = 1/2s(\psi)\dot{\psi}^2, \quad s(\psi) = (\alpha_{11}\alpha_{22} - \alpha_{12}^2)/\alpha_{11} > 0$$

and, by virtue of the equations of motion, $dE/dt = u\dot{\psi}$. For constant values of the control $u = \pm a$, the phase curves

$$1/2s(\psi)\dot{\psi}^2 = \pm a(\psi - \psi_f) + c$$

each have a common point with the straight line $\dot{\psi} = 0$. The branches of the curve with $c = 0$ constitute the line of switching of the optimal synthesis and the number of switch-overs is no greater than one. This can be shown in the coordinates

$$\dot{y} = \sqrt{s(\psi)}\dot{\psi}, \quad y = \int_{\psi_f}^{\psi} \sqrt{s(\theta)}d\theta$$

where, by virtue of the monotonicity of the function $y(\psi)$, the inverse function $\psi = e(y)$ exists and the equation of motion takes the form

$$\ddot{y} = \nu, \quad |\nu| \le a/\sqrt{s(e(y))}$$

The form of the curve (2.2) in the torus $\varphi \times \psi$ obviously depends on the mass-inertia parameters λ and μ of the two-link device. The possibility (during the controlled motion) of changing their values should enable one to pass from one curve to another, thereby linking the initial equilibrium position (φ_0, ψ_0) and any required final equilibrium position of the system (φ_f, ψ_f) by a common trajectory.

Example. Suppose a single instantaneous change in the parameter μ (for example, by "discharging of ballast" Δm_2) is permissible in the case of the robot-manipulator (Fig. 1) with an internal moment $|u| \le a$. Assuming that the values of $\mu_1 = m_1/m_2$ and $\mu_2 = m_1/(m_2 - \Delta m_2)$ are specified, we will determine whether it is possible for the system to pass in a finite time from an arbitrary position of rest $(\varphi_0, 0, \psi_0, 0)$ (when $\mu = \mu_1$) to any required position $(\varphi_f, 0, \psi_f, 0)$ (when $\mu = \mu_2$) in the cylinder $\mathbf{T}^2 \times \mathbf{R}^2$ by a single replacement of μ_1 by μ_2 $(\mu_1 < \mu_2)$.

In fact, the first and second segments of the required trajectory differ in the parameters $b_2 > b_1 > 1$ in Eq. (2.2) which can be represented in the form

$$\varphi - \Phi(\psi, b) = \text{const} \tag{2.5}$$

Assuming the number λ to be fixed, we will investigate two possible cases.

The case *($\lambda \le 1$).* In this case, the relation $\varphi(\psi)$ (2.2) is monotonic and the coefficient k (2.4) increases as b increases:

$$\frac{dk}{db} = \frac{b - \lambda}{\lambda\sqrt{b^2 - 1}^3} > 0 \quad (b > 1 \ge \lambda)$$

Consequently, for sufficiently large $n \in N$, the continuous function $[\Phi(\psi, b_1) - \Phi(\psi, b_2) + c]$ at the ends of the interval $\psi \in [-2n\pi, +2n\pi]$ takes values $c \pm [k(b_1) - k(b_2)]n\pi$ of different signs, that is, it vanishes for a certain value of ψ. Therefore, in the covering space $\mathbf{R}^2$ (and this means also in the torus $\mathbf{T}^2$), the curve $\varphi - \Phi(\psi, b_1) = \text{const}$ drawn through the point (φ_0, ψ_0) inevitably intersects the curve $\varphi - \Phi(\psi, b_2) = \text{const}$ drawn through the point (φ_f, ψ_f). We denote the coordinates of their point of intersection by (φ_*, ψ_*) (in the torus $\varphi \times \psi$, it may be non-unique). Then, from an initial state of rest $(\varphi_0, 0, \psi_0, 0)$ when $\mu = \mu_1$, it is possible according to Remark 1 to reach the state $(\varphi_*, 0, \psi_*, 0)$ (by means of an acceleration and retardation) and subsequently, when $\mu = \mu_2$, to pass (using a similar strategy) from the state $(\varphi_*, 0, \psi_*, 0)$ to the state $(\varphi_f, 0, \psi_f, 0)$.

The inevitability of the intersection of curves (2.5) when $b = b_1$ and $b = b_2$ also enables one (by replacing b_1 by b_2 at the intermediate position of rest $(\varphi_*, 0, \psi_*, 0)$) to pass in the case of an admissible control $|u(t)| \le a$ from any state $(\varphi_0, \dot\varphi_0, \psi_0, \dot\psi_0) \in \Gamma$ to any state $(\varphi_f, \dot\varphi_f, \psi_f, \dot\psi_f) \in \Gamma$ in a finite time.

This can be achieved, for example, in three stages. Having first calculated the values of φ_2, ψ_2 at the end of the "retardation" section

$$u_2(t): (\varphi_f, \dot\varphi_f, \psi_f, \dot\psi_f) \to (\varphi_2, 0, \psi_2, 0) \quad (u_2 = -\operatorname{sign}\dot\psi_f)$$

with the third stage we perform the "acceleration"

$$-u_2(t): (\varphi_2, 0, \psi_2, 0) \to (\varphi_f, \dot\varphi_f, \psi_f, \dot\psi_f)$$

At the end of the first stage

$$u_1(t): (\varphi_0, \dot\varphi_0, \psi_0, \dot\psi_0) \to (\varphi_1, 0, \psi_1, 0) \quad (u_1 = -\operatorname{sign}\dot\psi_0)$$

the quantities φ_1, ψ_1 become known and, in the second stage, it is therefore sufficient to join the states of rest $(\varphi_1, 0, \psi_1, 0)$ and $(\varphi_2, 0, \psi_2, 0)$ which have been found by a trajectory using the strategy considered above (with the intermediate replacement of the parameter μ_1 by μ_2).

The case *(when $\lambda > 1$).* In this case, when $b = \lambda$, the function $k(b)(-1 < k(b) < 0)$ has a unique minimum which is equal to $\sqrt{1 - 1/\lambda^2} - 1$. Infinitely many pairs of values $b_1 \neq b_2$ therefore exist for which $k(b_1) = k(b_2)$, that is, the two trajectories (2.5) (when $b = b_1$ and $b = b_2$) emerging from a common point (φ_0, ψ_0) in $\mathbf{R}^2$ again intersect at the point $(\varphi_0, \psi_0 + 2\pi)$. The width of the domain enclosed between these curves is defined in the interval $\psi \in [\psi_0, \psi_0 + 2\pi]$ as $\max_\psi|\nu|$, where $\nu = \Phi(\psi, b_1) - \Phi(\psi, b_2)$. From expressions (2.3) and (2.4), we obtain

$$\Phi(\psi, b_i) = \frac{1}{2}\psi + (k+1)\operatorname{arctg}\left(\chi_i \operatorname{tg}\frac{\psi}{2}\right), \quad \chi_i = \sqrt{\frac{b_i - 1}{b_i + 1}}, \quad i = 1, 2$$

The condition for an extremum $d\nu/d\psi = 0$ takes the form

$$\operatorname{tg}\frac{\psi}{2} = \sqrt{\frac{1}{\chi_1\chi_2}}, \quad \text{т.е.} \quad \max_\psi|\nu| = (k+1)\left|\operatorname{arctg}\sqrt{\frac{\chi_2}{\chi_1}} - \operatorname{arctg}\sqrt{\frac{\chi_1}{\chi_2}}\right| = \Delta$$

By taking the quantities b_1 and b_2 sufficiently close to one another (with the same condition $k(b_1) = k(b_2)$), we obtain a width Δ as small as desired. Then, the first trajectory $\varphi - \Phi(\psi, b_1) = \text{const}$ drawn through the point (φ_0, ψ_0) does not intersect the "second" trajectory $\varphi - \Phi(\psi, b_1) = \text{const}$ drawn through $(\varphi_0 + \Delta + \delta, \psi_0)$ (where $0 < \delta \ll 1$) in $\mathbf{R}^2$. On choosing (at the expense of b_1 and b_2) the value of k to be rational, it is possible to obtain a strip in the torus $\mathbf{T}^2$ between the curves which is closed and does not fill the torus.

In fact, for each rational number $k = -m/n$ $(0 < m < n;\ n, m \in N)$, it is possible to set up a corresponding range of values of λ, calculated from the equation

$$\sqrt{1 - 1/\lambda^2} - 1 = k - \varepsilon$$

with a variable small parameter ε $(0 \le \varepsilon < \varepsilon_k < k + 1)$, where the quantity ε_k will be formulated below as a function of λ. The function

$$\lambda(k, \varepsilon) = 1/\sqrt{1 - (k + 1 - \varepsilon)^2}$$

introduced in this way has a value of λ, when $\varepsilon = 0$, which corresponds to the conditions $b_1 = b_2$ and $\Delta = 0$. On expressing b_1 and b_2 (with the same condition $k(b_1) = k(b_2)$) in terms of k and $\lambda(k, \varepsilon)$ and substituting into the formulae for χ_i $(i = 1, 2)$, we obtain the function $\Delta(k, \varepsilon)$ which is monotonic (in the domain $0 < \Delta < \pi\,(1 + k)$) for all $k = \text{const}$. The function which is inverse to it reduces to the form

$$\varepsilon = (1 + k)[1 - \cos(\Delta/(1 + k))]$$

The requirement that $\Delta < 2\pi/n$ (for the number of turns n along the coordinate ψ considered) can be ensured by the condition

$$\varepsilon < \varepsilon_k, \quad \varepsilon_k = (n-m)[1-\cos(\pi/(n-m))]/n$$

On mapping the set of rational numbers $k \in]-1, 0[$ into the set of intervals $\lambda(k, \varepsilon)$ $(0 \le \varepsilon < \varepsilon_k)$ using the method described above, we obtain the unification of these intervals in the form of a domain $]1, \infty[$. Then, for each $\lambda_0 > 1$, suitable originals k_0 and ε exist for which the quantities b_1 and b_2 (which satisfy the condition $k(b_1) = k(b_2) = k_0$ ensure the inequality $\Delta < 2\pi/n_0$, i.e. the strip between the "first" and "second" curves, which is closed in the torus, will not have self intersections.

Values of the parameters μ_1 and μ_2 therefore exist (before and after "discharging ballast") for which a transition from the point (φ_0, ψ_0) to the point (φ_f, ψ_f) is impossible in the general case if $\lambda > 1$.

Thus, the transport problem of translating a planar two-link device from an arbitrary state of rest with a configuration (φ_0, ψ_0) to any specified state (φ_f, ψ_f) when $|u(t)| \le a$ by means of a single instantaneous change of the parameter μ_1 to μ_2 subject to the condition that $\lambda \le 1$, always has a solution but does not have a solution for certain values of μ_1 and μ_2 when $\lambda > 1$.

The controllability parameters (in the manifold $p_1 = 0$) can be considered as being parametrically conditional, since they depend very much on the mass-inertia parameters μ and λ.

3. A model of a two-component device with an additional degree of freedom

The idea of achieving controllability by varying the parameters of a "two-component device which is uncontrollable with respect to u" (Fig. 1) leads, for example, to the following modification of this object (Fig. 2).

Suppose it is possible to move a point mass m_3 on a spring without friction along the first link. By measuring its position by the coordinate x_3 from the end of the unstressed spring of length d, we obtain the distance from the fixed hinge up to the mass m_3 in the form $d + x_3$. The oscillations of the point m_3 will imitate the change in the mass-inertia parameters of the two-component device over a known range, due to which its controllability properties are "improved" (in the sense of the preliminary discussion). Note that the addition of a further pendulum (in a gravitational field) to the first hinge, which models for example, partial filling of an internal cavity in the first link with a liquid, would be an equivalent modification.

For simplicity, we will take a quadratic elasticity potential

$$B(x_3) = \frac{1}{2}c_3x_3^2$$

where c_3 is the stiffness of the spring. Retaining the previous notation (Section 2) for μ, λ, u, a, t, we introduce the additional dimensionless parameters

$$\mu_3 = \frac{m_3}{m_2}, \quad \delta = \frac{d}{l_2}, \quad x = \frac{x_3}{l_2}, \quad c = \frac{c_3l_2}{m_2g}$$

We then obtain

$$L = \frac{1}{2}\dot{\mathbf{q}}^T A(\mathbf{q})\dot{\mathbf{q}} - \frac{1}{2}cx^2, \quad \mathbf{q} = (\varphi, \psi, x)^T \in \mathbf{T}^2 \times \mathbf{R}$$

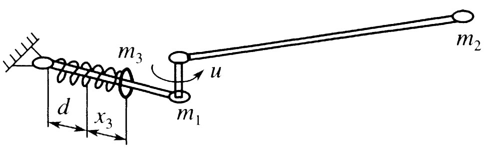

Fig. 2.

for the reduced Lagrangian function, where $\mathbf{q}$ is the configuration vector and the non-zero elements of the matrix $A(\mathbf{q})=\{\alpha_{ij}(\mathbf{q})\}$ $(i, j=1, 2, 3)$ have the form

$$\alpha_{11}(\psi, x) = (\mu+1)\lambda^2 + 1 + 2\lambda\cos\psi + \mu_3(\delta + x)^2$$

$$\alpha_{12}(\psi) = \alpha_{21}(\psi) = 1+\lambda\cos\psi, \quad \alpha_{22} = 1, \quad \alpha_{33} = \mu_3$$

We now write down the equations of motion of the system

$$p_1 = \alpha_{11}(\psi, x)\dot{\varphi} + \alpha_{12}(\psi)\dot{\psi} = \text{const}$$
$$(1+\lambda\cos\psi)\ddot{\varphi} + \ddot{\psi} = -\lambda\dot{\varphi}^2\sin\psi + u, \quad \ddot{x} + \Omega^2 x = \dot{\varphi}^2(\delta + x) \tag{3.1}$$

where $\Omega = \sqrt{c/\mu_3}$ is the frequency of natural oscillations of the load m_3 and p_1 is the angular momentum of the system about the axis of rotation of the first link; its constancy defines an invariant manifold in the phase space $\text{TM} = \mathbf{T}^2 \times \mathbf{R}^4$, in view of which the system is not globally controllable in TM using an input u.

We next consider the behaviour of the object in the manifold $\Gamma = \{\mathbf{q}, \dot{\mathbf{q}} : p_1 = 0\}$, which contains, in particular, all the possible equilibrium positions of the system. The aim of the subsequent discussion is to demonstrate the controllability of system (3.1) in the manifold Γ under the action of a moment $|u(t)| \le a$ (a is a specified constant), that is, the possibility of passing from any initial state $(\mathbf{q}_0, \dot{\mathbf{q}}_0) \in \Gamma$ to any required state $(\mathbf{q}_f, \dot{\mathbf{q}}_f) \in \Gamma$ in a finite time. Here, we restrict ourselves to the condition $\lambda < 1$, when the length of the second rod l_2 exceeds the length l_1 of the first rod. In particular, this enables us to describe the manifold Γ by the condition

$$\dot{\psi} = -\frac{\alpha_{11}(\psi, x)}{\alpha_{12}(\psi)}\dot{\varphi} \tag{3.2}$$

since it is guaranteed that $\alpha_{12}(\psi) = 1 + \lambda\cos\psi \neq 0$.

The dependence of the controllability properties on the numerical parameter λ was seen in Section 2. In particular, it was shown earlier in Ref. 3 that a planar two-link device outside a gravitational field is controllable by the action of just a single external moment $|u_1(t)| \le a$ (a is a specified constant) subject to the condition $\lambda \neq 1$. This exception was associated with the invariant manifold $\psi \equiv \pi$, when, for $l_1 = l_2$, the point m_2 and the fixed hinge are superposed.

Further, for system (3.1) when $\lambda < 1$, we shall rely on the possibility of particular solutions

$$\dot{\varphi} \equiv \omega, \quad x \equiv x_*, \quad \psi = \psi_*(t) \tag{3.3}$$

where the constants ω and x_* are related by the condition

$$\Omega^2 x_* = \omega^2(\delta + x_*); \quad \Omega^2 > \omega^2, \quad x_* > 0 \tag{3.4}$$

and the function $\psi_*(t)$ is determined from the first equation of (3.1)

$$\dot{\psi} = -G(\psi, x_*)\omega; \quad G(\psi, x) = \alpha_{11}(\psi, x)/\alpha_{12}(\psi) > 0 \tag{3.5}$$

Such a state can be ensured by the part u_1 of the control action $u = u_1 + u_2$ by calculating it from the second equation of (3.1) in the form

$$u_1 = \ddot{\psi} + \lambda\omega^2\sin\psi$$

At the same time, from the equality

$$u_1\dot{\psi} = \ddot{\psi}\dot{\psi} + \lambda\omega^2\dot{\psi}\sin\psi = \frac{d}{dt}\left(\frac{1}{2}\dot{\psi}^2 - \lambda\omega^2\cos\psi\right)$$

when account is taken of expression (3.5), we conclude that, since $\dot{\psi} \neq 0$ (when $\omega \neq 0$), then

$$u_1 = \omega^2 R(\psi, x_*); \quad R(\psi, x_*) = \frac{\partial}{\partial\psi}\left(\frac{1}{2}G^2(\psi, x_*) - \lambda\cos\psi\right) \tag{3.6}$$

($R(\psi, x_*)$ is a continuous function). In the interval $\psi \in [0; 2\pi]$, for each value of x_* (which depends monotonically on ω), we obtain

$$\max_\psi |R(\psi, x_*)| = R\left(\frac{1}{2}\pi, x_*\right) = \lambda[(\mu+1)\lambda^2 + \mu_3(\delta + x_*)^2]^2$$

It follows from expression (3.6) that the quantity $\max_\psi |u_1|$ depends monotonically on ω, that is, a value of ω_* exists such that, when $\omega \le \omega_*$, the constraint $|u_1| \le a/2$ is guaranteed. Then, under the conditions of the procedure (3.3)–(3.6), we assume

$$0 < \omega \le \omega_* \tag{3.7}$$

Consequently, when the first rod executes a uniform rotation and the load m_3 is in relative equilibrium, the particular solution (3.3) is possible for system (3.1) (subject to the condition that $\lambda < 1$) with the control (3.6). This regime is impossible when $\lambda \ge 1$.

The image point moves in the torus $\varphi \times \psi$ according to Eq. (2.2) if we take

$$b = [(\mu+1)\lambda^2 + 1 + \mu_3(\delta + x_*)^2]/(2\lambda) \tag{3.8}$$

In other words, in the case of relative equilibrium of the load m_3, the motion will be the same as in the case of a two-link pendulum without a spring (Section 2) but with a "reduced" parameter

$$\mu_* = \mu + \mu_3(\delta + x_*)^2/\lambda^2$$

The curve in the torus will be closed if x_* (together with ω) is chosen such that the number k in equality (2.4) turns out to be rational.

In order to investigate the controllability properties of system (3.1), we will use its dynamical subsystem (in ψ, $\dot\psi$, x, $\dot x$) obtained from condition (3.2). Then, the second momentum coordinate has the form

$$p_2 = \alpha_{12}(\psi)\dot\varphi + \alpha_{22}(\psi)\dot\psi = \rho\dot\psi; \quad \rho(\psi, x) = (\alpha_{11}\alpha_{22} - \alpha_{12}^2)/\alpha_{11} > 0$$

Eliminating $\dot\varphi$ in this way, we obtain the equations of motion

$$\frac{d}{dt}(\rho\dot\psi) = \frac{1}{2}\dot\psi^2\frac{\partial\rho}{\partial\psi} + u, \quad \ddot x + \Omega^2 x = \left(\frac{\dot\psi}{G}\right)^2(\delta + x) \tag{3.9}$$

where $G = G(\psi, x)$. Here, when $u \equiv 0$, there is an energy integral

$$E(\psi, \dot\psi, x, \dot x) = \frac{1}{2}\rho(\psi, x)\dot\psi^2 + \frac{1}{2}\mu_3\dot x^2 + \frac{1}{2}cx^2 \tag{3.10}$$

We will now show that it is possible to translate system (3.9) by means of an admissible control $|u| \le a$ (a is a specified constant) from an arbitrary initial state (ψ_0, $\dot\psi_0$, x_0, $\dot x_0$) to any required state (ψ_f, $\dot\psi_f$, x_f, $\dot x_f$) in a finite time.

In the case of systems which are reversible in time ($t \to -t$), global controllability is guaranteed in the case of null-controllability[1] when the object can be transferred, using an admissible control $u(t)$, from any initial state ($\mathbf{q}_0$, $\dot{\mathbf{q}}_0$) to the state ($\mathbf{0}$, $\mathbf{0}$). Here, the feasibility of the transition $(\mathbf{q}_0, \dot{\mathbf{q}}_0) \to (\mathbf{0}, \mathbf{0})$ would follow, for example, from a combination of local controllability in the neighbourhood of 0, 0 and stabilizability, that is, from the attainment of any ε-neighbourhood of the point ($\mathbf{0}$, $\mathbf{0}$) (see Assertion 1). A special feature of system (3.9) being considered is the fact that there is no local controllability in the neighbourhood of the equilibrium positions in the linear approximation. The following is therefore proposed.

Remark 2. The global controllability of system (3.9) is guaranteed, if the system

1) is locally controllable in the neighbourhood of the solution (3.3)–(3.5),
2) is stabilizable to this state, that is, it is transferred (by means of an admissible control $u(t)$) in a finite time from an arbitrary initial state to any ε-neighbourhood of the solution (3.3)–(3.5).

Actually, a combination of these conditions enables one to reach the manifold (3.3)–(3.5) from an arbitrary initial state $(\psi_0, \dot\psi_0, x_0, \dot x_0)$. In view of the symmetry $\dot{\mathbf{q}} \to -\dot{\mathbf{q}}$, $t \to -t$, the state $\dot\varphi \equiv -\omega$, $x \equiv x_*$, $\psi = -\psi_*(t)$ will also be accessible in the case of a natural Lagrangian system. Hence, together with the motion

$$u(t)\colon (0, 0, 0, 0) \to (\psi_1, -\dot\psi_1, x_*, 0)$$

(where $\dot\psi_1 = +\omega G(\psi_1, x_*)$), the "inverse" transition

$$u(T-t)\colon (\psi_1, \dot\psi_1, x_*, 0) \to (0, 0, 0, 0)$$

will also be possible.

Since the values of ψ_1, $\dot\psi_1$ are periodically repeated along the trajectory of the solution (3.3)–(3.5), the motion

$$(\psi_0, \dot\psi_0, x_0, \dot x_0) \to (\psi_1, \dot\psi_1, x_*, 0) \to (0, 0, 0, 0)$$

will be guaranteed, that is, global controllability will occur.

Further, to prove conditions 1 and 2 (see Remark 2), we set up the equation in the form

$$u = u_1 + u_2; \quad |u_1| \le a/2, \quad |u_2| \le a/2$$

We shall specify the function u_1 in advance in the form of the feedback $u_1(\psi, \dot\psi, x, \dot x)$ such that, in the manifold (3.3)–(3.5), it is identical to the function (3.6), and choose the "free" component u_2, guided by the aim of reaching the manifold (3.3)–(3.6). We now consider conditions 1 and 2 separately.

Condition 1. It is well known[1] that it is sufficient to discover local controllability in the linear approximations, that is, by linearizing system (3.9) in the neighbourhood of system (3.3)–(3.6).

Introducing the notation

$$\eta = \psi - \psi_*, \quad \beta = x - x_*$$

we obtain the following linear equations for the perturbations

$$\begin{aligned} \ddot\eta &= a_{21}(t)\eta + a_{22}(t)\dot\eta + a_{23}(t)\beta + a_{24}(t)\dot\beta + w \\ \ddot\beta &= a_{41}(t)\eta + a_{42}(t)\dot\eta + a_{43}(t)\beta \end{aligned} \tag{3.11}$$

where

$$a_{41}(t) = \frac{\nu}{G}\frac{\partial G}{\partial\psi}, \quad a_{42}(t) = \frac{\nu}{G\omega} < 0, \quad a_{43}(t) = \omega^2 - \Omega^2 - \frac{4\omega^2\mu_3(\delta + x_*)^2}{\alpha_{11}} < 0$$

$$\nu = -2\omega^2(\delta + x_*) < 0$$

$a_{2i}(t)$ are smooth functions of $\psi_*(t)$ (i=1, 2, 3, 4) and the functions $G(\psi, x)$, $\partial G/\partial\psi$, α_{11} (ψ, x) are calculated using relations (3.3)–(3.5).

The new control $w = u_2/\rho(\psi_*(t), x_*)$ is "equivalent" to the old control u_2 from the point of view of controllability: since the function $\rho > 0$ has an upper limit, the existence of an appropriate function $w(t)$ also guarantees the existence of the required function $u_2(t)$.

Introducing the vector $\mathbf{z} = (\eta, \dot\eta, \beta, \dot\beta)^T$, we can represent system (3.1) in the form

$$\dot{\mathbf{z}} = \Lambda(t)\mathbf{z} + \mathbf{B}w; \quad \Lambda(t) = \begin{Vmatrix} 0 & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & 1 \\ a_{41} & a_{42} & a_{43} & 0 \end{Vmatrix}, \quad \mathbf{B} = \begin{Vmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Vmatrix} \tag{3.12}$$

It is well known[8] that the existence of an instant of time t_* (in the finite interval of time of the motion being considered) for which rank $\mathbf{K}(t_*) = 4$ is a sufficient condition for the linear unsteady system (3.12) be the controllable. The controllability matrix $\mathbf{K} = (\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3, \mathbf{K}_4)$ is composed of the columns

$$\mathbf{K}_1(t) = \mathbf{B}, \quad \mathbf{K}_i(t) = \Lambda(t)\mathbf{K}_{i-1}(t) - d\mathbf{K}_{i-1}(t)/dt, \quad i = 2, 3, 4$$

In the given case, the determinant of the matrix $\mathbf{K}$ is simplified by subtracting from some columns a linear combination of the other columns. After expanding the determinant, for all t we obtain

$$\det K = -\det\begin{Vmatrix} a_{42} & h - \dot{a}_{42} \\ h & a_{42}a_{43} - \dot{h} \end{Vmatrix} = -a_{42}^2 a_{43} > 0 \tag{3.13}$$

that is, rank $\mathbf{K} = 4$, and this means that there is local controllability with respect to the input u_2 in the case of system (3.9) in the neighbourhood of system (3.3)–(3.6).

In writing out the last link in the chain of equalities (3.13), we have taken into account the fact that

$$h = a_{41} - \dot{a}_{42} = \frac{\nu}{G}\frac{\partial G}{\partial\psi} - \frac{d}{dt}\left(\frac{\nu}{\omega G}\right) \equiv 0$$

since

$$\frac{dG}{dt} = \frac{\partial G}{\partial\psi}\dot{\psi} = -\omega G\frac{\partial G}{\partial\psi}$$

along the “reference” solution (3.3)–(3.6).

Condition 2. We will use the direct Lyapunov method to prove stabilizability and consider the function

$$V = \frac{1}{2}\rho\xi^2 + \frac{1}{2}\mu_3\dot{\beta}^2 + \frac{1}{2}\mu_3(\Omega^2 - \omega^2)\beta^2; \;\; \rho = \rho(\psi, x_* + \beta), \;\; \zeta = \dot{\psi} + G(\psi, x)\omega \tag{3.14}$$

Taking account of the energy integral (3.10), we write the derivative of the function (3.14), by virtue of the equations of motion (3.9), in the form

$$\frac{dV}{dt} = \left(u + \frac{\omega}{2G}\frac{dU}{dt}\right)(\dot{\psi} + G\omega), \quad U = \rho(\psi, x)G^2(\psi, x) - \mu_3(x + \delta)^2 \tag{3.15}$$

where the property

$$G^2\partial\rho/\partial x = 2\mu_3(x + \delta)$$

is used in the course of the transformations.

In the space TM of the vectors $(\psi, \xi, \beta, \dot{\beta})$, the sets $M_c = \{\psi, \xi, \beta, \dot{\beta} : V \leq c\}$ are compact for any value of ω.

The subspace of the vectors $\mathbf{y} = (\xi, \beta, \dot{\beta})$ is denoted by TM_1. Then, in view of the easily verified property

$$\rho(\psi, x) \geq \rho(0, x), \;\; \forall\psi$$

we have $V(\psi, \mathbf{y}) \geq V(0, \mathbf{y}) = W(\mathbf{y})$, where $W(\mathbf{y})$ is a connected Lyapunov function[6] in TM_1. In particular, it follows from the equality $W = 0$ that $\mathbf{y} = \mathbf{0}$. Hence, in order to establish asymptotic stability in the large in the sets from TM, it is sufficient[5,9] to prove that, by virtue of Eq. (3.9) (by an appropriate choice of the control $u(t)$), we obtain $dV/dt \leq 0$ and that, apart from $\mathbf{y} = \mathbf{0}$, there are no integral trajectories in the set $dV/dt \equiv 0$.

If the first component of the control $u = u_1 + u_2$ is taken in the form

$$u_1 = -\frac{\omega}{2G}\left(\frac{\partial U}{\partial\psi}\dot{\psi} + \frac{\partial U}{\partial x}\dot{x}\right) \tag{3.16}$$

then, in the objective set (3.3)–(3.5) (where $\dot{x} \equiv 0$, $\dot{x} \equiv x_*$, $\dot{\psi} \equiv -G\omega$), this function is identical to the function (3.6) since

$$R(\psi, x_*) = \frac{1}{2}\frac{\partial}{\partial\psi}[\rho(\psi, x_*)G^2(\psi, x_*)]$$

Eq. (3.15) takes the form $dV/dt = u_2\xi$. This means that it is sufficient to choose a smooth function u_2 which is bounded (from the condition that $|u_2| \le a/2$) and opposite in sign to ξ (for example, $u_2 = -(a/\pi)$ arctg ξ) in order to obtain $dV/dt \le 0$.

Here, the condition $dV/dt = 0$ only arises when $\xi = 0$, that is, $\dot{\psi} \equiv -G\omega$ which is only possible along the solution (3.3)–(3.6), where $\mathbf{y} = \mathbf{0}$.

Hence, the control which has been found $u = u_1 + u_2$ will ensure stabilizability. In order simultaneously to satisfy the constraint $|u_1| \le a/2$ in the case of the function u_1 (3.16), the value of ω can be chosen to be "fairly small", depending on the actual initial state $(\mathbf{q}_0, \dot{\mathbf{q}}_0)$ (where $\mathbf{q} = (\psi, x)^T$) for which it is required to prove the feasibility of the transition $(\mathbf{q}_0, \dot{\mathbf{q}}_0) \to (\mathbf{0}, \mathbf{0})$.

The procedure for choosing ω is as follows:

a) $\max_\omega V(\mathbf{q}_0, \dot{\mathbf{q}}_0) = c_1$ is calculated for a specific vector $0 \le \omega \le \omega_*$ in the compactum $(\mathbf{q}_0, \dot{\mathbf{q}}_0)$ for the smooth Lyapunov function (3.14),
b) $c = \max\{c_1, c_2\}$ is chosen, where $c_2 = \max_\omega V(\mathbf{0}, \mathbf{0})$,
c) the value of

$$\max_{(\mathbf{q},\dot{\mathbf{q}})}\left|\frac{1}{2G}\dot{\mathbf{q}}^T\frac{\partial U}{\partial \mathbf{q}}\right| = \sigma$$

is found in the compactum $M_c = \{\psi, \xi, \beta, \dot{\beta} : V \le c\}$, where the function $U(\psi, x)$ is taken in the form of (3.15),
d) finally,

$$\omega_1 = \max\{a/(2\sigma), \omega_*\}$$

is chosen.

The constraint $|u| \le a$ is thereby guaranteed for a control $u = u_1 + u_2$ with components in the form

$$u_1 = -\frac{\omega_1}{2G}\frac{dU}{dt}, \quad u_2 = -\frac{a}{\pi}\text{arctg}(\dot{\psi} + G\omega_1)$$

This control transfers the system from the state $(\mathbf{q}_0, \dot{\mathbf{q}}_0)$ to any ε-neighbourhood of the solution (3.3)–(3.6) when $\omega = \omega_1$. The control giving a transition from the state $(\mathbf{0}, \mathbf{0})$ to the neighbourhood of the "symmetric" system (for which $\omega = -\omega_1$) will also be bounded.

In view of the fact that $\omega_1 \ne 0$, condition (3.13) is satisfied in both neighbourhoods, that is, there is local controllability. The motion $(\mathbf{q}_0, \dot{\mathbf{q}}_0) \to (\mathbf{0}, \mathbf{0})$ therefore exists (see Remark 2). The null-controllability of subsystem (3.9) which has been proved is equivalent to its global controllability, that is, to the possibility of transferring the subsystem by means of an admissible control $|u| \le a$ in a finite time from any state $(\psi_0, \dot{\psi}_0, x_0, \dot{x}_0)$ to any required state $(\psi_f, \dot{\psi}_f, x_f, \dot{x}_f)$.

Remark 3. It can be shown that, when $\lambda < 1$ (that is, the second link is longer than the first), system (3.1) with three degrees of freedom is globally controllable in the manifold $p_1 = 0$. This object (Fig. 2) can be transferred, for example, from any position of rest to any required state (with zero angular momentum p_1) by means of an admissible control $|u| \le a$ (a is a specified constant) in a finite time.

The proof follows in Section 6 (Example 2).

Note that the controllability (in the manifold $p_1 = 0$) of the "modified" model of a two-link device, which has been revealed, holds for as small a value of the mass m_3 of the additional load as desired whereas, when $m_3 = 0$, it was absent. This suggests that there are possible qualitative differences in the controllability properties of mechanical models of one and the same actual object when their details are described to different extents. Hence, in certain cases, by taking account of a small change in the parameters (the insignificant variability of a “rigid body”, the flexibility of a link in a construction, etc.) one can ensure the controllability of a mechanism which would not be controllable on the assumption that the links were absolutely rigid. In Section 4, this property will be called parametric controllability.

4. The concept of parametric controllability

Suppose that, in a dynamical control system, the state vector is specified in a finite-dimensional phase space TM and consists of two parts: in the form of state vectors $\mathbf{y} \in \mathrm{TM}_1$ (of the simplified model; dim $\mathbf{y} = n$) and $\mathbf{z} \in \mathrm{TM}_2$ (of additional coordinates, refining the model). The control vector $\mathbf{u}$ is chosen from a specified bounded domain $U \subset \mathbf{R}^m$, where $2m \le n$ in the case of the mechanical systems being discussed. The small parameter $\varepsilon > 0$ characterizes the closeness of the two models.

If the velocities of the additional degrees of freedom tend to zero when $\varepsilon \to 0$, the “refined” object can have the form of a regularly perturbed system

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \mathbf{z}, \varepsilon, \mathbf{u}), \quad \dot{\mathbf{z}} = \varepsilon \mathbf{F}(\mathbf{y}, \mathbf{z}, \varepsilon), \quad \mathbf{u} \in U \tag{4.1}$$

If, however, these velocities tend to infinity when $\varepsilon \to 0$, the system

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \mathbf{z}, \varepsilon, \mathbf{u}), \quad \varepsilon \dot{\mathbf{z}} = \mathbf{F}(\mathbf{y}, \mathbf{z}, \varepsilon), \quad \mathbf{u} \in U \tag{4.2}$$

will be singularly perturbed. In both cases, the simplified model (when $\varepsilon = 0$)

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \mathbf{z}_0, 0, \mathbf{u}), \quad \mathbf{u} \in U \tag{4.3}$$

which symbolizes, for example, the absolute rigidity of the bodies (compared with their small variability), can be set to correspond to the object (4.1) (or (4.2)). Here, the solutions of the systems being compared for values of the parameters $\varepsilon = 0$ and $1 \gg \varepsilon > 0$ are assumed to be close, that is, we assume that the conditions of Poincaré’s theorem are guaranteed in the case of system (4.1) and the conditions of Tikhonov’s theorem[10] in the case of (4.2) (where $\mathbf{z} = \mathbf{0}$ corresponds to the equality $\mathbf{F}(\mathbf{y}, \mathbf{z}, 0) = \mathbf{0}$, that is, $\mathbf{z}_0 = \mathbf{0}$ is chosen in Eq. (4.3)).

The controllability properties of the “flexible” model (4.1) (or (4.2)) and the “rigid” model (4.3) may differ.

Definition. We say that system (4.1) (or (4.2)) is parametrically controllable in the set $P_1 \subset \mathrm{TM}_1$ if, in the case of a $\varepsilon > 0$ which may be as small as desired, it is possible to transfer the system from any state $\mathbf{y}_0 \in P_1$ to any other required state $\mathbf{y}_f \in P_1$ by means of an admissible control in a finite time and it is impossible to do this in the general case when $\varepsilon = 0$.

The controllability property which has been introduced is considered to be “parametric” in the same sense as, for example, parametric resonance in the oscillation theory, that is, it is caused by an “unfreezing” the parameters. The additional equations (for $\mathbf{z}$), which distinguish the “flexible” system from the “rigid” system, actually describe the change in the mass-inertia characteristics which, when $\varepsilon = 0$, take constant values.

Note that parametric controllability is not identical to “weak controllability”[10] since, in the general case, the vector $\mathbf{u}$ is not multiplied by the coefficient ε. Systems (4.1) (or (4.2)) are only “weakly controllable” in a figurative sense, since the controls are bounded by quantities which may be as small as desired. Among these units, “parametrically controllable” units are additionally distinguished not because of the absence of an input action in the simplified model (when $\varepsilon = 0$) but in view of the properties of the intrinsic dynamics: a “rigid” system is controllable in the case of actions which may be as large as desired and a “flexible” system is controllable in the case of actions which may be as small as desired.

5. The sufficient conditions for the parametric controllability of system (1.1) with a cyclic coordinate

According to the definition of parametric controllability (Section 4), its recognition implies a comparison of two close models of a mechanical unit of which the “rigid” model (with the smaller number of degrees of freedom) will be uncontrollable, unlike the “flexible” model. The cause of the uncontrollability of the rigid model could be, for example, the existence of a cyclic coordinate (we shall call it q_1) for which $\partial L/\partial q_1 = 0$, $a_1 = 0$. The conservation law $p_1 = \partial L/\partial \dot{q}_1 = \text{const}$ defines the invariant manifold $\Gamma \subset \text{TM}$. We will now consider this case in greater detail, representing the vector $\mathbf{q}$ in system (1.1) in the form $\mathbf{q} = (q_1, \mathbf{x}^T)^T$, where $\mathbf{x} = (q_2, q_3, \ldots, q_n)$ such that the inertia matrix A and the potential B will depend solely on $\mathbf{x}$. Using Routh's function $R = L - \dot{q}_1 \partial L/\partial \dot{q}_1$, we obtain the dynamical subsystem

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{\mathbf{x}}}\right) - \frac{\partial R}{\partial \mathbf{x}} = \mathbf{u}_x, \quad \mathbf{x} \in \mathbf{R}^{n-1}; \quad \mathbf{u} = (0, \mathbf{u}_x^T)^T, \quad \mathbf{u}_x \in U_x \subset \mathbf{R}^{n-1} \tag{5.1}$$

in the space $(\mathbf{x}, \dot{\mathbf{x}})$.

In this case, the set of states of rest will be:

$$\zeta_x = \{(\mathbf{x}, \dot{\mathbf{x}}): \dot{\mathbf{x}} = \mathbf{0}, \partial B/\partial \mathbf{x} = \mathbf{0}, \mathbf{u}_x = \mathbf{0}\}$$

The case when $p_1 = 0$, which corresponds to motion with an initial (or final) equilibrium position, is of practical importance. We will next ascertain under which conditions system (1.1) will be controllable in this manifold $\Gamma \subset \text{TM}$, that is, it can be transferred from any state $(\mathbf{q}_0, \dot{\mathbf{q}}_0) \in \Gamma$ to any other required state $(\mathbf{q}_f, \dot{\mathbf{q}}_f) \in \Gamma$ in a finite time.

We separate out the first row of the $n \times n$ inertia matrix $A(\mathbf{x})$, eliminating the left (first) element from it, dividing the remaining part of the row by it, and denoting the resulting $(n-1)$-vector by $\boldsymbol{\sigma}(\mathbf{x}) = (\sigma_1, \sigma_2, \ldots, \sigma_{n-1})^T$. The invariant manifold $p_1 = 0$ is then described by the equation

$$\dot{q}_1 + \boldsymbol{\sigma}^T(\mathbf{x})\dot{\mathbf{x}} = 0 \tag{5.2}$$

For a rigid system, the configurational vector $(q_1, \mathbf{s}^T)^T$ will have a dimensional $k < n$ and each coordinate of the vector $\mathbf{s}$ is encountered among the coordinates of the vector $\mathbf{x}$. The remaining $n - k$ coordinates of the vector $\mathbf{x}$ correspond to additional degrees of freedom and take constant values on changing from the flexible model to the rigid model for which the law of conservation of momentum is written (by analogy with Eq. (5.2)) in the form

$$\dot{q}_1 + \boldsymbol{\eta}^T(\mathbf{s})\dot{\mathbf{s}} = 0 \tag{5.3}$$

where the vector $\boldsymbol{\eta}(\mathbf{s}) = (\eta_1, \eta_2, \ldots, \eta_{k-1})^T$ is obtained from the first row of the $(k \times k)$ inertia matrix of the rigid system using the same method as for $\boldsymbol{\sigma}(\mathbf{x})$.

The controllability properties of the systems in the zero momentum manifold Γ (which is defined, in the case of the rigid model, in the form (5.3) and, in the case of the flexible model, in the form (5.2)) may differ and depend, as will be shown, on the properties of the matrices $G = \partial\boldsymbol{\sigma}/\partial\mathbf{x}$ and $N = \partial\boldsymbol{\eta}/\partial s$.

Assertion 2. Suppose system (1.1) with a small parameter ε has a cyclic coordinate q_1 to which the conservation law (5.2) corresponds (when $\varepsilon > 0$) or the conservation law (5.3) corresponds (when $\varepsilon = 0$). At the same time, suppose that, in the case of the dynamical subsystem (5.1), the domain of null-controllability is identical to the whole of the phase space $(\mathbf{x}, \dot{\mathbf{x}})$. Then, if $N^T = N$, $G^T \neq G$, system (1.1) is parametrically controllable in the manifold (5.2).

Proof. We will demonstrate the controllability of the flexible system (5.1), (5.2), that is, the feasibility of a transition from an arbitrary point $(q_{10}, \dot{q}_{10}, \mathbf{x}_0, \dot{\mathbf{x}}_0)$ to any required point $(q_{1f}, \dot{q}_{1f}, \mathbf{x}_f, \dot{\mathbf{x}}_f)$ if both satisfy condition (5.2).

Actually, by virtue of the global controllability of subsystem (5.1), an admissible control

$$\mathbf{u}_1(t): (\mathbf{x}_0, \dot{\mathbf{x}}_0) \rightarrow (\mathbf{0}, \mathbf{0})$$

can be found to which the motion $(q_{10}, \dot{q}_{10}, \mathbf{x}_0, \dot{\mathbf{x}}_0) \rightarrow (r_1, 0, \mathbf{0}, \mathbf{0})$ in the phase space TM of system (1.1) corresponds with a certain finite value r_1 of the coordinate q_1. The controllable transition

$$\mathbf{u}_3(t): (q_{1f}, -\dot{q}_{1f}, \mathbf{x}_f, -\dot{\mathbf{x}}_f) \rightarrow (r_2, 0, \mathbf{0}, \mathbf{0}), \quad t \in [0, T_3]$$

can also be found to which the "symmetric" motion

$$\mathbf{u}_3(T_3 - t): (r_2, 0, \mathbf{0}, \mathbf{0}) \rightarrow (q_{1f}, \dot{q}_{1f}, \mathbf{x}_f, \dot{\mathbf{x}}_f)$$

corresponds in inverse time $(t > -t)$.

It remains to demonstrate the feasibility of the transition

$$\mathbf{u}_2(t)\colon (r_1, 0, \mathbf{0}, \mathbf{0}) \to (r_2, 0, \mathbf{0}, \mathbf{0})$$

or of the transition which is equivalent to it (by virtue of the cyclic nature of the coordinate q_1)

$$\mathbf{u}_2(t)\colon (0, 0, \mathbf{0}, \mathbf{0}) \to (r_2 - r_1, 0, \mathbf{0}, \mathbf{0})$$

To be specific, we will put $r_2 - r_1 = \Delta r > 0$ and, in the phase space TM of system (1.1), we will consider the attainability set $K(T)$[1] from the point $(0, 0, \mathbf{0}, \mathbf{0})$ during a time T, which is chosen to be sufficiently short in order that the domain of the corresponding motions when $t \in [0, T]$ should be simply-connected. The projection of $K(T)$ onto the phase subspace $(\mathbf{x}, \dot{\mathbf{x}})$ will be a non-empty set (by virtue of the controllability of subsystem (5.1)) Since the set of admissible controls is convex, the set $K(T)$ is compact, convex and depends continuously on T (Ref. 1, p. 78). Its boundary $\partial K(T)$ intersects the line $l = \{\dot{q}_1 = 0, \mathbf{x} = \mathbf{0}, \dot{\mathbf{x}} = \mathbf{0}\}$ at the point with the coordinate $q_1 = r$.

The inequality $r \neq 0$, which is proved indirectly, holds.

We shall assume that the coordinate q_1 is equal to zero at the end of each trajectory starting at the point $(0, 0, \mathbf{0}, \mathbf{0})$ and ending on the line l. By virtue of the local controllability of system (5.1), a cone of controls with a dimension of 2 $(n-1)$ at the initial point $(\mathbf{0}, \mathbf{0})$ will correspond to the set of such trajectories in the space $(\mathbf{x}, \dot{\mathbf{x}})$. Writing Eq. (5.2) in the form $dq_1 = -\sigma^T(\mathbf{x})d\mathbf{x}$, we obtain

$$r = -\oint \sigma_1 dx_1 + \sigma_2 dx_2 + \dots + \sigma_{n-1} dx_{n-1} = 0$$

for any closed contour $(\mathbf{0}, \mathbf{0}) \to (\mathbf{x}, \dot{\mathbf{x}}) \to (\mathbf{0}, \mathbf{0})$. This means that, in the simply-connected domain of motions being considered, the vector field $(\sigma_1, \sigma_2, \ldots, \sigma_{n-1})$ is potential (Ref. 11, p. 335). The system of equalities

$$\partial\sigma_i/\partial x_j = \partial\sigma_j/\partial x_i, \quad \forall i, j = 1, 2, \ldots, n-1$$

which is equivalent to the equality $G^T = G$, which contradicts the condition, follows from this. This means that $r \neq 0$.

In view of the symmetry of system (1.1) with respect to $t \to -t$, to be specific, we can put $r > 0$.

Since the set $K(T)$ depends continuously on T, a $T_* \leq T$ and a certain controlled motion

$$\mathbf{u}(t)\colon (0, 0, \mathbf{0}, \mathbf{0}) \to (r_*, 0, \mathbf{0}, \mathbf{0}), \quad t \in [0, T_*]$$

is found for each value of r_* (where $0 \leq r_* \leq r$). This enables us to make the transition

$$\mathbf{u}_2(t)\colon (0, 0, \mathbf{0}, \mathbf{0}) \to (\Delta r, 0, \mathbf{0}, \mathbf{0})$$

after $m+1$ stages (where $m \leq \Delta r/r \leq m+1$), during which the first m will be of the same duration T and the last one takes a time T_*, corresponding to the magnitude of $r_* = \Delta r - mr$. Thus, a motion

$$(q_{10}, \dot{q}_{10}, \mathbf{x}_0, \dot{\mathbf{x}}_0) \to (r_1, 0, \mathbf{0}, \mathbf{0}) \to (r_2, 0, \mathbf{0}, \mathbf{0}) \to (q_{1f}, \dot{q}_{1f}, \mathbf{x}_f, \dot{\mathbf{x}}_f)$$

is found in the space TM which proves the controllability of the flexible system (1.1) in the manifold (5.2). By virtue of the equality $N^T = N$, the conditions for the vector field $(\eta_1, \eta_2, \ldots, \eta_{k-1})$ to be a potential field

$$\partial\eta_i/\partial s_j = \partial\eta_j/\partial s_i, \quad \forall i, j = 1, 2, \ldots, k-1$$

are satisfied here.

The existence of the potential function $F(q_1, \mathbf{s}) = 0$ makes the configurations $(q_{1f}, \mathbf{s}_f)$, for which $F(q_{1f}, \mathbf{s}_f) \neq 0$, inaccessible. This means that the rigid system (5.3) is not controllable in the manifold (5.3) as is required.

Note that a link between the controllability of a dynamical system and the non-integrability of an auxiliary differential equation, obtained by the projection of a vector field, has been described earlier in Ref. 12. The difference from the proposed Assertion 2 lies in the fact that the non-integrability of Eq. (5.2) in such a case[12] would be a necessary but not a sufficient condition. Hence, the main burden in establishing the controllability of system (1.1) falls on the special properties of a natural Lagrangian system: satisfying the conditions of Assertion 1 replaces verification of the non-integrability of a certain multidimensional dynamical subsystem.

We also note that, in the case of Assertion 2, only an "angular" coordinate can occur as the cyclic coordinate q_1 in a real mechanical system since, in the case of a "distance" q_1, the condition $p_1 = 0$ is usually integrated (in terms of the "motion of the centre of mass of the system").

6. Examples of singularly perturbed systems (4.2)

Example 1. We will now show that the system (Fig. 2) considered in Section 3 is parametrically controllable in the invariant manifold Γ of zero angular momentum if it is assumed that $\varepsilon = \mu_3$. Since the global controllability of the dynamical subsystem (3.9) has already been proved in Section 3, it remains (by virtue of Assertion 2) to consider the matrices $G = \partial\boldsymbol{\sigma}/\partial\mathbf{x}$ and $N = \partial\boldsymbol{\eta}/\partial s$. In this case,

$$x_1 = \psi, \quad x_2 = x, \quad \boldsymbol{\sigma} = [\alpha_{12}(x_1)/\alpha_{11}(x_1, x_2), 0]^T, \quad \boldsymbol{\eta} = [\alpha_{12}(x_1)/\alpha_{11}(x_1, x_2)]$$

Hence

$$\partial\sigma_1/\partial x_i \neq 0, \quad \partial\sigma_2/\partial x_i = 0, \quad i = 1, 2$$

whence it follows that $G^T \neq G$. Since the matrix N of the rigid system is a 1×1 matrix, we have $N^T = N$. According to Assertion 2, the system (Fig. 2) is parametrically controllable (in the mentioned manifold Γ) if $m_3 = 0$ (Fig. 1) corresponds to the rigid system.

Example 2. Suppose that, in the planar two-link pendulum being considered, there is again a unique control, that is, a bounded internal moment u which acts "from the first rod to the second rod". We will assume that $|u| \leq a$, where a is a specified constant. In addition to the rigid model (in which the links OA and AB are assumed to be absolutely rigid), a flexible model is considered in the simplest form when the second rod AB is made up from two absolutely rigid parts (with lengths l_2 and l_3), which are joined by a cylindrical hinge with a spiral spring of high stiffness c (Fig. 3). The first rod OA (of length l_1) is assumed to be an absolutely rigid body in both models. For simplicity, we will assume that all the links are weightless and that their masses m_1, m_2 and m_3 are concentrated in the hinges. The flexible model has a configuration $\mathbf{q} = (q_1, \mathbf{x}^T)^T$, $\mathbf{q} \in \mathbf{T}^2 \times \mathbf{R}$), where q_1 is the angle of the first rod measured from the fixed axis $\mathbf{x} = (x_1, x_2)^T, \mathbf{x} \in T \times R$, x_1 is the angle of the second rod measured from the first rod, and x_2 is the angle of elastic deviation (deformation of the spring) from the unstressed axis of the second bar.

In the rigid model, the vector of the generalized coordinates $(q_1, s_1)^T \in T^2$, where $s_1 = x_1$.

We now consider the motion of the flexible system, introducing the dimensionless parameters and dimensionless time

$$\mu_i = \frac{m_i}{m_1}, \quad \lambda_i = \frac{l_i}{l_1} \ (i = 2, 3), \quad a' = \frac{a}{m_1 g l_1}, \quad u' = \frac{u}{m_1 g l_1}, \quad c' = \frac{c l_1}{m_1 g}, \quad t' = t\sqrt{\frac{g}{l_1}}$$

(we will henceforth omit the primes). In the reduced Lagrangian function

$$L = \frac{1}{2}\dot{\mathbf{q}}^T A(\mathbf{x})\dot{\mathbf{q}} - B(\mathbf{x})$$

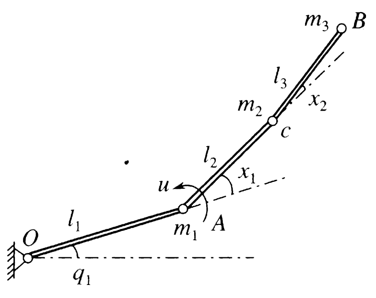

Fig. 3.

the elastic potential energy $B(\mathbf{x}) = 1/2cx_2^2$. The positive definite, symmetric 3 × 3 inertia matrix $A(\mathbf{x})$ has the elements

$$\begin{aligned} a_{11} &= a_{22} + h_1 + 2h_2\cos x_1 + 2h_3\cos(x_1 + x_2) \\ a_{12} &= a_{22} + h_2\cos x_1 + h_3\cos(x_1 + x_2), \quad a_{13} = a_{23} + h_3\cos(x_1 + x_2) \\ a_{23} &= h_3\lambda_3 + h_3\lambda_2\cos x_2, \quad a_{22} = h_2\lambda_2 + h_3\lambda_3 + 2h_3\lambda_2\cos x_2, \quad a_{33} = h_3\lambda_3 \end{aligned} \tag{6.1}$$

where

$$h_1 = 1 + \mu_2 + \mu_3, \quad h_2 = \lambda_2(\mu_2 + \mu_3), \quad h_3 = \lambda_3\mu_3$$

The coordinate q_1 is cyclic, and the angular momentum about the axis of rotation of the first rod

$$p_1 = \partial L/\partial \dot{q}_1 = \text{const}$$

is therefore conserved.

Next, we consider the case when $p_1 = 0$, which corresponds to an initial (or final) state of rest. Condition (5.2), where $n = 3$, $\sigma = (a_{12}, a_{13})^T/a_{11}$ and $a_{11}(\mathbf{x}) > 0$, is satisfied in this invariant manifold. Routh's function is represented in the form

$$R = \frac{1}{2}\dot{\mathbf{x}}^T\rho(\mathbf{x})\dot{\mathbf{x}} - B(\mathbf{x})$$

and the elements of the symmetric, positive definite 2 × 2 matrix $\rho(\mathbf{x}) = \{\rho_{ij}(\mathbf{x})\}$ are expressed in terms of the cofactors A_{ij} of the elements a_{ij} of the matrix $A(\mathbf{x})$:

$$\rho_{11} = A_{33}/a_{11}, \quad \rho_{12} = -A_{23}/a_{11}, \quad \rho_{22} = A_{22}/a_{11} \tag{6.2}$$

The equations of motion of the flexible model therefore include the kinematic condition (5.2) and the dynamical subsystem (5.1), where

$$\mathbf{u}_x = \mathbf{b}u, \quad \mathbf{b} = (1, 0)^T, \quad |u| \le a$$

and a is a specified number.

We will now demonstrate the global controllability of this system in the space $(\mathbf{x}, \dot{\mathbf{x}})$, relying on Assertion 1. In the neighbourhood of an arbitrary state of rest

$$\dot{\mathbf{x}} = \mathbf{0}, \quad \mathbf{x} = \mathbf{x}_0, \quad \mathbf{x}_0 = (x_1, 0)^T, \quad u = 0$$

the linearized equation of motion in the notation $y = \rho(x_0)(x - x_0)$ takes the form

$$\ddot{\mathbf{y}} = \mathbf{D}\mathbf{y} + \mathbf{b}u; \quad \mathbf{D} = -\mathbf{B}_0\rho^{-1}(\mathbf{x}_0), \quad \mathbf{B}_0 = \text{diag}(0, c) \tag{6.3}$$

For the controllability of system (6.3), it is sufficient[8] that the condition

$$\text{rank}\mathbf{K} = 2; \quad \mathbf{K} = (\mathbf{b}, \mathbf{Db}), \quad \det\mathbf{K} = c\rho_{12}(\mathbf{x}_0)/\det\rho(\mathbf{x}_0); \quad c/\det\rho(\mathbf{x}_0) > 0$$

is satisfied.

It follows from relations (6.1) and (6.2) that

$$\begin{aligned} \rho_{12} = h_3[&(h_1 + h_2\cos x_1)(\lambda_3 + \lambda_2\cos x_2) - \\ &- \cos(x_1 + x_2)(h_2\lambda_2 + h_2\cos x_1 + h_3\lambda_2\cos x_2 + h_3\cos(x_1 + x_2))]/a_{11} \end{aligned} \tag{6.4}$$

Hence, the condition $\det\mathbf{K} > 0$ is guaranteed, for example, when $\cos x_1 = 0$, that is, in the neighbourhoods of the states of rest

$$x_1 = \pi/2 + k\pi, \quad k \in Z, \quad x_2 = 0, \quad \dot{\mathbf{x}} = \mathbf{0} \tag{6.5}$$

(with mutually perpendicular rods OA and AB) the flexible model is locally controllable.

The stabilizability of the system to an equilibrium state (6.5) when $k=0$ is ensured by the component $u_1 = -(a/2)\sin(x_1 - \pi/2)$ of the control $u = u_1 + u_2$, $|u_2| \leq a/2$. On adding u_1 to the potential forces, we obtain the reduced potential

$$B_1(\mathbf{x}) = B(\mathbf{x}) + a(1 - \cos(x_1 - \pi/2))/2$$

as a connected Lyapunov function[6] in the space $\mathbf{T}^1 \times \mathbf{R}^1$ of the variables $(x_1 - \pi/2)$ and x_2, and all of the sets $B_1(\mathbf{x}) \leq c_1$ $(c_1 > 0)$ will be compact.

We will now show that the equations of motion (5.1) and (5.2) when $u_2 \equiv 0$ and the actions of the "potential force" u_1 do not allow of the particular solution $\dot{x}_1 \equiv 0$ (excluding the equilibrium positions (6.5)).

In fact, suppose $\dot{x}_1 \equiv 0$. Then, the case when $\sin(x_1 - \pi/2) \equiv \text{const} \neq 0$ is impossible since, as a consequence of the condition $u_1 \equiv \text{const} \neq 0$, it would be at variance with the equilibrium between the moments of all of the forces, including d'Alembert's inertial force, with respect to the point A.

Assuming that $\sin(x_1 - \pi/2) \equiv 0$, we obtain $u_1 \equiv 0$. Then, in the free motion (when $u_2 \equiv 0$), the potential energy $B(\mathbf{x})$ will be bounded, and the instant of time $t = t_1$ is therefore found at which $\dot{x}_2 = 0$, and this means that $\dot{q}_1 = 0$ (in view of the assumption that $\dot{x}_1 \equiv 0$ and by virtue of (5.2)). Hence, when $t = t_1$, we obtain an instantaneous state of rest $\dot{\mathbf{q}} = 0$ for which some of the components in Lagrange's equations disappear:

$$A(\mathbf{x})\ddot{\mathbf{q}} + B_1\mathbf{q} = 0; \quad \mathbf{B}_1 = \operatorname{diag}(0, 0, c)$$

The first two equations of this system are then written in the form

$$A_1\ddot{\mathbf{z}} = \mathbf{0}; \quad \mathbf{z} = (q_1, x_1)^T, \quad \det A_1 = (a_{11}a_{23} - a_{13}a_{21}) = a_{11}\rho_{12}(\mathbf{x})$$

Note that, when the values

$$x_1 = \pi/2 + k\pi, \quad k \in Z$$

are substituted into the right-hand side of equality (6.4), the function $\rho_{12} = f(x_2)$ will be even. The total energy of the system is constant and, in the states $\dot{\mathbf{q}} = 0$, is equal to the elastic potential energy. This means that the maximum and minimum deformations x_2 in the instantaneous states of rest differ solely in sign and, for one of them, it is found that $f(x_2) \neq 0$, that is, $\det A_1 \neq 0$. In this state, we not only obtain that $\dot{\mathbf{z}} = \mathbf{0}$ but, also, $\ddot{\mathbf{z}} = \mathbf{0}$ which corresponds to a condition of rest of the whole system. Consequently, the particular solution $\dot{x}_1 \equiv 0$ is only possible in the equilibrium positions (6.5). According to Assertion 1, the flexible system (5.1), (5.2) is globally controllable in the space $(\mathbf{x}, \dot{\mathbf{x}})$.

Asymmetry of the matrix $G = \partial\boldsymbol{\sigma}/\partial\mathbf{x}$, that is, satisfaction of the condition

$$\frac{\partial}{\partial x_2}\left(\frac{a_{12}}{a_{11}}\right) \neq \frac{\partial}{\partial x_1}\left(\frac{a_{13}}{a_{11}}\right)$$

can be observed by direct calculations even in the linear approximation in the neighbourhood of the configuration $x_1 = \pi/2$, $x_2 = 0$. In this case, the matrix $N = \partial\boldsymbol{\eta}/\partial\mathbf{s}$ for the rigid system is a 1×1 matrix. Consequently, the conditions of Assertion 2 are satisfied in the example considered, that is, there is parametric controllability in the manifold $p_1 = 0$. When account is taken of elastic pliability, the two-link pendulum (Fig. 3) can be transferred by means of a bounded internal moment in a finite time from any state of rest into any other required state of rest.

Note that Example 2 corresponds precisely to the case of a singularly perturbed system (4.2), which clearly manifests itself in the equations by making the replacement $x_2 = \varepsilon\theta$, where $\varepsilon = 1/c$ is a small parameter.

7. Example of a regularly perturbed system (4.1)

Suppose the system (Fig. 4) with three degrees of freedom and a configuration $\mathbf{q} = (\varphi_1, \varphi_2, \varphi_3)^T$ consists of continuous homogeneous discs (of mass m_2 and m_3 and with radii R_2 and R_3) and a pole OA (a homogeneous rod of mass m and length l). The construction is located in the horizontal plane (outside a gravitational field) and allows of slipping of the discs. All the angles are measured anticlockwise. There is no friction at the hinges O and A, and the Amonton-Coulomb law is obeyed at the point K where the discs touch: $F_f \leq fN$, where we assume that N is a known constant. The control action is an internal moment $M(t)$ which acts from the rod OA to the disc with its centre at A. In the investigation

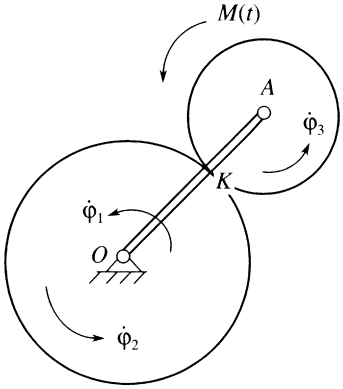

Fig. 4.

of the controllability properties, we put $|M(t)| \le M_0$, where M_0 is a prespecified quantity which is less than all the subsequently calculated "characteristic" moments in the dynamical system.

When there is no slipping, the coupling force $F_a \le fN$ ensures the kinematic relation

$$\dot\varphi_1 l = \dot\varphi_2 R_2 + \dot\varphi_3 R_3$$

If, however, the slip rate

$$\dot s = \dot\varphi_2 R_2 + \dot\varphi_3 R_3 - \dot\varphi_1 l$$

is non-zero, the friction slip force will be

$$F_f = fN \operatorname{sign}\dot s \tag{7.1}$$

Note that the condition $\dot s \equiv 0$ (motion without slipping) will be the result of the "infinitely fast" alternation of the value of the function $\operatorname{sign}(\dot s)$. At the same time, the function, which is calculated using Filippov's rule[13] and is equivalent to $fN\operatorname{sign}(\dot s(t))$, is identical to the Coulomb coupling force $F_a \le fN$. Hence, in the general case, it is possible to compare the equations of motion, assuming there is slipping, by taking the friction force in the form of (7.1) beforehand. Using the dimensionless variables and dimensionless time

$$\mu_i = \frac{m_i}{m_3}\ (i = 1, 2), \quad \lambda = \frac{R_2}{l} < 1, \quad s_1 = \frac{s}{l}, \quad u = \frac{M}{m_3 g l}, \quad \eta = \frac{Nf}{m_3 g}, \quad t_1 = t\sqrt{\frac{g}{l}}$$

we obtain the reduced kinetic energy

$$T = \frac{1}{2}\left(\frac{\mu_1}{3} + 1\right)\dot\varphi_1^2 + \frac{1}{4}\mu_2\lambda^2\dot\varphi_2^2 + \frac{1}{4}(1-\lambda)^2\dot\varphi_3^2$$

and the equations of motion

$$\begin{aligned} &\left(\frac{\mu_1}{3} + 1\right)\ddot\varphi_1 + \frac{1}{2}(1-\lambda)^2\ddot\varphi_3 = \lambda\eta\operatorname{sign}\dot s_1 \\ &\frac{1}{2}\mu_2\lambda^2\ddot\varphi_2 = -\lambda\eta\operatorname{sign}\dot s_1, \quad \frac{1}{2}(1-\lambda)^2\ddot\varphi_3 = u - (1-\lambda)\eta\operatorname{sign}\dot s_1 \end{aligned} \tag{7.2}$$

where

$$\dot s_1 = -\dot\varphi_1 + \lambda\dot\varphi_2 + (1-\lambda)\dot\varphi_3$$

Since there are no external moments about the axis O, the angular momentum

$$p_1 = \left(\frac{\mu_1}{3} + 1\right)\dot\varphi_1 + \frac{1}{2}\mu_2\lambda^2\dot\varphi_2 + \frac{1}{2}(1-\lambda)^2\dot\varphi_3$$

is conserved, that is, the system remains all the time in the invariant manifold $p_1 = \text{const}$. We then investigate the controllability with respect to an input $u(t)$, keeping φ_1 and s_1 as the independent variables. Introducing the notation

$$\omega = \left(\frac{\mu_1}{3}+1\right)\dot{\varphi}_1, \quad \nu = \frac{\dot{s}_1}{c_1}, \quad k = \frac{c_2}{c_1}, \quad c_1 = \frac{2}{1-\lambda}+\frac{3}{\mu_1+3}, \quad c_2 = \frac{2(\mu_2+1)}{\mu_2}+\frac{3}{\mu_1+3}$$

in the manifold $p_1 = \text{const}$, we obtain the motion of a system with two degrees of freedom

$$\dot{\omega} = -u+\eta\,\text{sign}\nu, \quad \dot{\nu} = u-k\eta\,\text{sign}\nu \tag{7.3}$$

where ω and ν are the reduced angular velocity of the rod OA and the slip rate of the discs. We impose a specific constraint $|u| \le a$ on the control, assuming it to be as small as desired including when $a < k\eta/2$.

Two types of motion are possible in the case of system (7.3):

A. In the case of an initial slip rate $\boldsymbol{\nu}(0) \neq 0$, the object passes in a finite time into a state $\boldsymbol{\nu} = 0$ since, by virtue of system (7.3), the derivative of the function $V = |\boldsymbol{\nu}|$ with respect to time will be

$$dV/dt = \dot{\nu}\,\text{sign}\nu = u\,\text{sign}\nu - k\eta < -k\eta/2$$

B. By virtue of the inequality $a < k\eta$, when $\boldsymbol{\nu}(0) = 0$, conditions for the infinitely rapid alternation of the sign of the function sign $\boldsymbol{\nu}$ arise, since $\boldsymbol{\nu} > 0 \Rightarrow \dot{\boldsymbol{\nu}} < 0, \boldsymbol{\nu} > 0 \Rightarrow \dot{\boldsymbol{\nu}} < 0$. At the same time, the equivalent right-hand side of the second equation of (7.3), calculated using a well-known procedure,[13] corresponds to a friction coupling force $ku(t)$ which doe not exceed the maximum force according to the Amonton-Coulomb law.

As a result, within the manifold $p_1 = \text{const}$ there is an invariant submanifold which is specified by the condition $\dot{s} = 0$ (rolling without slipping) which the system cannot abandon for any admissible control. Hence, the system is not controllable in the space $(\dot{\varphi}_1, \dot{s})$, and this also means in the space $(\dot{\varphi}_1, \dot{\varphi}_2)$.

We next complicate the model of a planetary mechanism (Fig. 4) by making the mutual pressure between the discs dependent on the angular velocity of the rod OA (in view of its longitudinal deformability, for example). Suppose $\eta = \eta(\omega) > 0$, where the function $\eta(\omega)$ is a smooth, even and monotonically decreasing function when $\omega > 0$ and an asymptotically decreasing function: $\omega > 0$ when $\omega \to 0$. In the case of small deformability of the rod, We can assume that $\dot{\eta} = \varepsilon f(\omega)\dot{\omega}$, where $d\eta/d\omega = \varepsilon f(\omega)$ and ε is a small parameter. Then, the flexible model

$$\dot{\omega} = -u+\eta\,\text{sign}\nu, \quad \dot{\nu} = u-k\eta\,\text{sign}\nu, \quad \dot{\eta} = \varepsilon f(\omega)(-u+\eta\,\text{sign}\nu) \tag{7.4}$$

which corresponds to a regularly perturbed system of the form of (4.1), can be treated together with the rigid model (7.3) (where $\eta = \text{const}$). Strictly speaking, this correspondence is only formal if the relation $\eta = \eta(\omega)$ is represented as a “first integral” in a system with an additional degree of freedom η, assuming that the different initial values $\eta(0)$ (due to precompression of the deformable rod OA) are constructively possible.

We will now investigate the controllability of system (7.4) in the space of the variables $(\omega, \boldsymbol{\nu})$ when $\varepsilon > 0$. As in the case of system (7.3), the relation

$$k\dot{\omega}+\dot{\nu} = (1-k)u \tag{7.5}$$

holds here which, when $k = 1$, gives the equality $\dot{\omega}+\dot{\boldsymbol{\nu}} = 0$ for any $u(t)$. It can be shown that the case $m_3R_3 = m_2R_2$ corresponds to the condition $k = 1$. With such a combination of parameters, systems (7.3) and (7.4) are not controllable in the manifold $p_1 = \text{const}$.

We next consider the case when $k < 1$, that is, when $m_3R_3 < m_2R_2$. Integrating system (7.4) for the value $u = +a$ separately in the half-planes $\boldsymbol{\nu} > 0$ and $\boldsymbol{\nu} < 0$, we obtain the set of solutions which is represented in Fig. 5. Here, by virtue of the properties of the function $\eta(\omega)$, numbers $b_2 > b_1 > 0$ exist such that $\eta(b_2) = a$, $\eta(b_1) = a/k$. On “splicing together” the solutions on the boundary $\boldsymbol{\nu} = 0$ of the half-planes inside the interval $|\omega| < b_1$ (where the inequality $a < k\eta(\omega)$ is satisfied), the conditions for a “sawtooth” state ($\boldsymbol{\nu} > 0 \Rightarrow \dot{\boldsymbol{\nu}} < 0, \boldsymbol{\nu} < 0 \Rightarrow \dot{\boldsymbol{\nu}} > 0$) are obtained, that is, a state of pure rolling.

For the case when $u = -a$, a family of solutions in the $(\omega, \boldsymbol{\nu})$ plane is obtained from a family of solutions shown in Fig. 5 by replacing the direction of the ω and ν-axes by the opposite axes since system (7.4) is invariant under

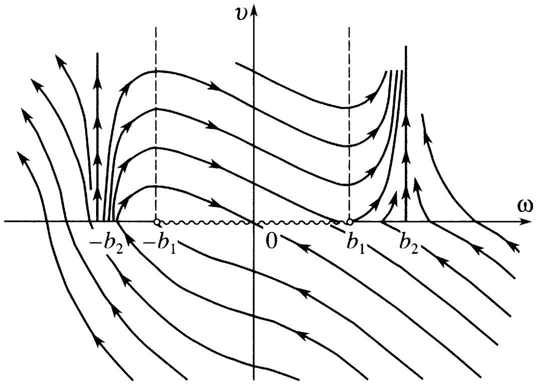

Fig. 5.

the replacement $u \to -u$, $\omega \to -\omega$, $\boldsymbol{\nu} \to -\boldsymbol{\nu}$. If the constraint a is replaced by a smaller value, the behaviour of the solutions is not qualitatively changed. Only the abscissae b_2 and b_1 of the characteristic verticals increase, that is, the range over which rolling occurs is extended and there is an increase in the angular velocity ω under steady-state conditions.

Note that, for the global controllability of system (7.4) in the space $(\omega, \boldsymbol{\nu})$, the null-controllability[1] of system (7.4) and the null-controllability of the time-reversed $t \to -t$ system in the form

$$\dot{\omega} = u - \eta(\omega)\text{sign}\nu, \quad \dot{\nu} = -u + k\eta(\omega)\text{sign}\nu \tag{7.6}$$

is sufficient.

In fact, any two points (ω_0, ν_0) and (ω_f, ν_f) can then be joined by a trajectory $(\omega_0, \nu_0) \to (0, 0) \to (\omega_f, \nu_f)$.

The null-controllability of system (7.4) is proved, for example, by the following strategy for the motion from an arbitrary point (ω_0, ν_0). If $\nu_0 \neq 0$, then, when $u = -a\text{sign}\nu$, the system passes after a finite time into a state $\nu = 0$ since, by virtue of system (7.4), the derivative of the function $V_1 = |\nu|$ with respect to time will have the form

$$dV_1/dt = \dot{\nu}\text{sign}\nu = -a - k\eta(\omega) < -a$$

If, however, $\nu_0 = 0$, then, from the point $(\omega_0, 0)$, it is possible to reach the point (0,0) after a finite time by means of a control

$$u = -a_1\text{sign}\omega, \quad 0 < a_1 < \min\{a, k\eta(\omega_0)\}$$

In view of the conditions for a "sawtooth" state, here

$$\dot{\nu} = 0, \quad \dot{\omega} = (1/k - 1)u$$

By virtue of system (7.4), for the function $V_2 = |\omega|$ we therefore obtain

$$dV_2/dt = -a_1(1/k - 1) \tag{7.7}$$

The null-controllability of the time-reversed system (7.6) is proved by the following method of moving from an initial point (ω_f, ν_f). If $\nu_f = 0$, it is possible to reach the point (0, 0) after a finite time by choosing a control

$$u = a_1\text{sign}\omega, \quad 0 < a_1 < \min\{a, k\eta(\omega_f)\}$$

Since $\dot{\nu} = 0$ follows from the equality $\dot{\omega} = -u(1/k - 1)$ and Eq. (7.6), then

$$\dot{\omega} = -a_1(1/k - 1)\text{sign}\omega.$$

So equality (7.7) is satisfied. Henceforth, time reversal of the "sawtooth" state is carried out by a preliminary replacement of the function sign(·) (with an infinitely rapid alternation of the sign) by an equivalent function using Filippov's rule[13].

If $\nu_f \neq 0$, then a transition in the set $\nu = 0$ has to be accomplished in a different manner for the cases $|\omega_f| > b_2$ and $|\omega_f| \leq b_2$.

The case *($|\omega_f| > b_2$).* Choosing a control $u = \eta(\omega_f)$ signν, we transfer a point in the (ω, ν) plane along the vertical $\omega = \omega_f$ (sign $\dot{\omega} = 0$) on the straight line $\nu = 0$. In view of the equality $\dot{\nu} = -(1-k)\eta(\omega)\text{sign}\nu$, we obtain

$$dV_1/dt = \dot{\nu}\text{sign}\nu = -(1-k)\eta(\omega)$$

Having reached the straight line $\nu = 0$, we now use the strategy $u = a_1$signω.

The case *($|\omega_f| \geq b_2$).* By choosing a control $u = -a$signν_f, (for which $\dot{\omega} = -(a + \eta(\omega))\text{sign}\nu$), we reach the vertical $\omega = -b_2$ (if $\nu_f > 0$) or $\omega = b_2$ (if $\nu_f < 0$). Here,

$$\dot{\nu} = (a + k\eta(\omega))\text{sign}\nu$$

i.e. the coordinate ν does not change sign. Having reached the vertical $|\omega| = b_2$, it is possible to move along it when $u = a$sign ν and, then, along the straight line $\nu = 0$ when $u = a_1$signω to the point (0,0).

Thus, when $k < 1$ (that is, when $m_3R_3 < m_2R_2$), it is possible to transfer system (7.4) from any state (ω_0, ν_0) to any other state (ω_f, ν_f) after a finite time by choosing an admissible control $|u| \leq a$, where a is a number which may be as small as desired. Unlike system (7.3) (where η = const.) controllability here is ensured by the possibility of emerging from a state of pure rolling into a state of slipping of the discs due to a change in the parameter η (ω).

It can be shown that, when $k > 1$ (that is, when $m_3R_3 < m_2R_2$), system (7.4) is uncontrollable. The family of its solutions when $u = +a$ (Fig. 6) differs from the family for $k < 1$ (Fig. 5) in the order of the arrangement of the verticals $|\omega| = b_1$ and $|\omega| = b_2$ (where $\eta(b_2) = a$, $\eta(b_1) = a/k$) since $b_2 < b_1$ already by now. As before, the family of solutions for $u = -a$ differs from the family for $u = +a$ by the replacement $\omega \to -\omega$, $\nu \to -\nu$. On the one hand, there is null-controllability here, that is, the attainment of the point (0,0) after a finite time from an arbitrary state (ω_0, ν_0) (using the same strategy as in the case when $k < 1$). On the other hand, from the point (0,0) it is only possible to reach a point on the ω axis, and points of the sector BAC (Fig. 6) or points of the sector which is symmetric to it about the centre 0. Actually, by virtue of system (7.4), motion from the point (0,0) is only possible under "sawtooth" conditions. When $u = +a$, it will reach the point $\omega = -b_1$ and, when $u = a_3 < a$, it continues "more to the left" (to the point $\omega = -b_3$, where $\eta(b_3) = a_3/k$). At the same time, BAC is the boundary of an oriented manifold,[14] that is, motion from its points can only occur along the sector or along the boundary. At each point of the curve AB, the normal vector $\mathbf{n} = (a - k\eta, a - \eta)^T$ makes a non-acute angle with the velocity vector $\boldsymbol{\nu} = (-u(t) + \eta, u(t) - k\eta)^T$ (since $\mathbf{n}\cdot\boldsymbol{\nu} = \eta(k-1)(u-a) \leq 0$ for any control $u(t)$). This means that system (7.4) is not globally controllable when $k \geq 1$.

Hence, when $k < 1$, system (7.4) is parametrically controllable in the space (ω, ν). Consequently, subject to the condition $m_3R_3 < m_2R_2$, the planetary mechanism (Fig. 4), using an internal moment $M(t)$ which may be as small as desired, can be transferred in the space ($\dot{\varphi}_1$, $\dot{\varphi}_2$) from any state to any other state due to the longitudinal flexibility of the rod OA, which changes the character of the slipping of the discs. This is made impossible when the rod OA is absolutely rigid.

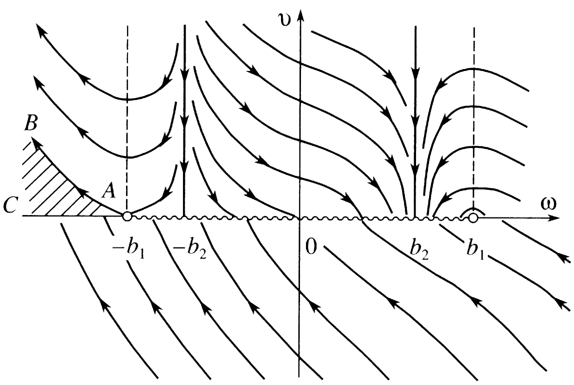

Fig. 6.

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